MATH 303 – Measure Theory
Homework 2

Please upload a pdf of your solutions by 23:59 on Monday, October 6. The assignment will be graded out of 10 points, taking into account both correctness and quality of presentation. More details on grading, as well as guidelines for mathematical writing, can be found on Moodle.

Problem 1. The goal of this problem is to gain a deeper understanding of integration in the context of the *counting measure*. Let X be a set, and let $a : X \to \mathbb{C}$. We say a is *unconditionally summable* to $c \in \mathbb{C}$, written

$$\sum_{x \in X} a(x) = c,$$

if for every $\varepsilon > 0$, there exists a finite set $F \subseteq X$ such that if $S \subseteq X$ is another finite set and $F \subseteq S$, then

$$\left| \sum_{x \in S} a(x) - c \right| < \varepsilon.$$

(a) Suppose $a : X \to \mathbb{C}$ is unconditionally summable. Prove that for any bijection $\sigma : X \to X$,

$$\sum_{x \in X} a(x) = \sum_{x \in X} a(\sigma(x)).$$

(b) Prove that a sequence $(a_n)_{n \in \mathbb{N}}$ is unconditionally summable[1] to $c \in \mathbb{C}$ if and only if

$$\sum_{n=1}^{\infty} |a_n| < \infty$$

and

$$\sum_{n=1}^{\infty} a_n := \lim_{N \to \infty} \sum_{n=1}^{N} a_n = c.$$

(You may cite the Riemann rearrangement theorem without reproving it.)

(c) Suggest a definition for unconditional summability in the extended real numbers (allowing for summability to ∞ or $-\infty$).

(d) Suppose now that $a : X \to [0, \infty]$ takes nonnegative values. Show that

$$\sum_{x \in X} a(x) = \int_X a \, d\#,$$

where $\# : \mathscr{P}(X) \to [0, \infty]$ is the counting measure on X and the sum on the left hand side is the unconditional sum in the extended real numbers.

(e) Suppose $a : X \to [0, \infty]$ and $\sum_{x \in X} a(x) < \infty$. Prove that $\{x \in X : a(x) > 0\}$ is countable.

[1] Here, we take $X = \mathbb{N}$ as the underlying space on which a is defined.

Solution: **(a)** Let $\sigma : X \to X$ be a bijection, and let $c = \sum_{x\in X} a(x)$. Let $\varepsilon > 0$ be given. By definition of unconditional summability of a, let $F \subseteq X$ be a finite set with the property that if $S \subseteq X$ is finite and $F \subseteq S$, then

$$\left|\sum_{x\in S} a(x) - c\right| < \varepsilon.$$

Put $F' = \sigma^{-1}(F)$. Suppose $S' \subseteq X$ is a finite set with $S' \supseteq F'$. Put $S = \sigma(S')$ so that

$$\sum_{x\in S'} a(\sigma(x)) = \sum_{x\in S} a(x).$$

But $S \supseteq \sigma(F') = F$, so

$$\left|\sum_{x\in S'} a(\sigma(x)) - c\right| = \left|\sum_{x\in S} a(x) - c\right| < \varepsilon.$$

This proves that $a \circ \sigma$ is unconditionally summable to c as desired.

(b) Suppose $(a_n)_{n\in\mathbb{N}}$ is unconditionally summable to c. We will first establish that the series $\sum_{n=1}^{\infty} a_n$ converges and then that it does so absolutely. Let $\varepsilon > 0$. By the definition of unconditional summability, let $F \subseteq \mathbb{N}$ be a finite set such that if $S \subseteq \mathbb{N}$ is finite and $F \subseteq S$, then

$$\left|\sum_{n\in S} a_n - c\right| < \varepsilon.$$

Let $N_0 = \max(F)$. If $N \geq N_0$, then $\{1, \ldots, N\} \supseteq F$, so

$$\left|\sum_{n=1}^{N} a_n - c\right| < \varepsilon.$$

Therefore, the series $\sum_{n=1}^{\infty} a_n$ converges to c. Suppose for contradiction that $\sum_{n=1}^{\infty} |a_n| = \infty$ so that the convergence is conditional. Then by the Riemann rearrangement theorem, there exists a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = c + 1.$$

But by (a), we must have $\sum_{n=1}^{\infty} a_{\sigma(n)} = \sum_{n=1}^{\infty} a_n = c$, so we have reached a contradiction. Hence, $\sum_{n=1}^{\infty} |a_n| < \infty$.

Conversely, suppose that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely to c. That is, $\sum_{n=1}^{\infty} |a_n| < \infty$ and $\sum_{n=1}^{\infty} a_n = c$. We will show that $(a_n)_{n\in\mathbb{N}}$ is unconditionally summable to c. Let $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} |a_n| < \infty$, there exists $N_1 \in \mathbb{N}$ such that

$$\sum_{n=N_1+1}^{\infty} |a_n| < \frac{\varepsilon}{2}.$$

Using the convergence of the series $\sum_{n=1}^{\infty} a_n$, there exists $N_2 \in \mathbb{N}$ such that if $N \geq N_2$, then

$$\left| \sum_{n=1}^{N} a_n - c \right| < \frac{\varepsilon}{2}.$$

Put $N = \max\{N_1, N_2\}$. Let $F = \{1, \ldots, N\}$. Then by the triangle inequality, if $S \subseteq \mathbb{N}$ is a finite set with $F \subseteq S$,

$$\left| \sum_{n\in S} a_n - c \right| \leq \left| \sum_{n\in S} a_n - \sum_{n=1}^{N} a_n \right| + \left| \sum_{n=1}^{N} a_n - c \right| \leq \sum_{n\in S\setminus F} |a_n| + \frac{\varepsilon}{2} \leq \sum_{n=N+1}^{\infty} |a_n| + \frac{\varepsilon}{2} < \varepsilon.$$

Thus, $(a_n)_{n\in\mathbb{N}}$ is unconditionally summable to c.

(c) The definition for unconditional summability in $\mathbb{C}$ uses ε-neighborhoods, since $\mathbb{C}$ is a metric space. We defined the topology on $[-\infty, \infty]$ by specifying a basis of open sets (the open intervals in $\mathbb{R}$ together with infinite rays of the form $(a, \infty]$ and $[-\infty, b)$) rather than a metric, so it is more natural to phrase convergence in $[-\infty, \infty]$ using open neighborhoods.

Say that a function $a : X \to [-\infty, \infty]$ is *unconditionally summable* to $c \in [-\infty, \infty]$ if for every open neighborhood U of c, there exists a finite set $F \subseteq X$ such that if $S \subseteq X$ is another finite set and $F \subseteq S$, then

$$\sum_{x\in S} a(x) \in U.$$

(d) Let us first prove that a is unconditionally summable in the extended real numbers. Let $c = \sup\{\sum_{x\in F} a(x) : F \subseteq X \text{ finite}\}$. Let $U \subseteq [0, \infty]$ be an open neighborhood of c. There exists an open interval[a] I such that $c \in I \subseteq U$. By the definition of the supremum, let $F \subseteq X$ be a finite set such that $\sum_{x\in F} a(x) \in I$. Given any finite set $S \subseteq X$ with $F \subseteq S$, since a is nonnegative, we have

$$\sum_{x\in F} a(x) \leq \sum_{x\in S} a(x) \leq c,$$

so

$$\sum_{x\in S} a(x) \in I \subseteq U.$$

By the definition of unconditional summability given in (c), we conclude that a is unconditionally summable to c.

Now let us check that the integral $\int_X a \; d\#$ is also equal to c. Given a finite set $F \subseteq X$, then function $a \cdot \mathbb{1}_F$ is a simple function that can be expressed as $a \cdot \mathbb{1}_F = \sum_{x\in F} a(x) \cdot \mathbb{1}_{\{x\}}$, so

$$\int_X a \cdot \mathbb{1}_F \; d\# = \sum_{x\in F} a(x).$$

Therefore, by the definition of the integral, $c \leq \int_X a \; d\#$. Let $0 \leq s \leq a$ be an arbitrary simple function. If $S = \{x \in X : s(x) \neq 0\}$ is finite, then $s \leq a \cdot \mathbb{1}_S$, so $\int_X s \; d\# \leq \sum_{x\in S} a(x) \leq c$. On the other hand, if S is infinite, then taking $t = \min_{x \in S} s(x) > 0$, we have that for finite sets $F \subseteq S$,

$$\sum_{x \in F} a(x) \geq t \cdot \#F,$$

so $c \geq \sup\{t \cdot \#F : F \subseteq S \text{ finite}\} = \infty$. In either case, $\int_X s \, d\# \leq c$. Taking the supremum over all simple functions $0 \leq s \leq a$, we conclude $\int_X a \, d\# \leq c$.

(e) For each $n \in \mathbb{N}$, let $E_n = \{x \in X : a(x) \geq \frac{1}{n}\}$. By (d) and Markov's inequality (Exercise 3),

$$\#E_n \leq n \cdot \sum_{x \in X} a(x) < \infty.$$

Therefore, $\{x \in X : a(x) > 0\} = \bigcup_{n \in \mathbb{N}} E_n$ is a countable union of finite sets, so it is countable.

[a] By interval, we mean either a traditional open interval (a, b) or an infinite ray $(a, \infty]$ in the case $c = \infty$.